

Thermodynamic Geometry and Locally Anisotropic Black Holes

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Thermodynamic properties of locally anisotropic $(2+1)$ -black holes are studied by applying geometric methods. We consider a new class of black holes with a constant in time elliptical event horizon which is imbedded in a generalized Finsler like spacetime geometry induced from Einstein gravity. The corresponding thermodynamic systems are three dimensional with entropy S being a hypersurface function on mass M , anisotropy angle θ and eccentricity of elliptic deformations ε . Two-dimensional curved thermodynamic geometries for locally anisotropic deformed black holes are constructed after integration on anisotropic parameter θ . Two approaches, the first one based on two-dimensional hypersurface parametric geometry and the second one developed in a Ruppeiner–Mrugala–Janyszek fashion, are analyzed. The thermodynamic curvatures are computed and the critical points of curvature vanishing are defined.

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I. INTRODUCTION

This is the second paper in a series in which we examine black holes for spacetimes with generic local anisotropy (la). In the first paper (hereafter referred to as paper I) we analyzed the low-dimensional locally anisotropic gravity (in brief, we shall use terms like la-gravity, la-spacetime, la-geometry, black la-holes and so on) and constructed new classes of locally anisotropic $2+1$ -black hole solutions [29].

In particular, it was shown following [26–28] how black holes can recast in a new fashion in generalized Finsler–Kaluza–Klein spaces and emphasized that such type solutions can be considered in the framework of usual Einstein gravity on anholonomic manifolds. We discussed the physical properties of $(2+1)$ -dimensional black holes with la-matter, induced by a rotating null fluid and by an inhomogeneous and non-static collapsing null fluid, and examined the vacuum polarization of la-spacetime by non-rotating black holes with ellipsoidal horizon and by rotating black la-holes with time oscillation and ellipsoidal horizons. We concluded that a general approach to the black la-holes should be based on a kind of nonequilibrium thermodynamics of such objects imbedded into la-spacetime ether being a continuous with possible dislocations and declinations. Nevertheless, we proved that for the simplest type of black la-holes their thermodynamics could be defined in the neighborhoods of some equilibrium states when the horizons are deformed but constant with respect to a frame base locally adapted to the nonlinear connection structure which model the local anisotropy.

In this paper we will specialize to the geometric thermodynamics of, for simplicity non-rotating, black la-

holes with elliptical horizons. We follow the notations and results from paper I (see also the Appendix) which are reestablished in a manner compatible in the locally isotropic thermodynamic [7] and spacetime [1] limits with the Banados–Teitelboim–Zanelli (BTZ) black hole. We defer the examination of higher dimension and string locally anisotropic black holes to a third paper (III) [28]. Here we also remark that a paper under preparation (IV) [30] is devoted to the nonequilibrium thermodynamics of black la-holes. This new approach (to black hole physics) is possible for la-spacetimes and is based on classical results [11,17,18,24].

We emphasize a postulate which is assumed (in non-explicit form) in general relativity: **the locally anisotropic distributed matter gives rise to a locally isotropic geometry**. This is contained in the structure of Einstein equations for a metric $g_{ij}(x^k)$ on a (pseudo) Riemannian space:

$$\begin{array}{c}
 G_{ij}(x^k) \simeq \\
 \boxed{\text{the Einstein tensor for a locally isotropic spacetime}} \\
 \Updownarrow \\
 T_{ij}(x^k, y^a), \\
 \boxed{\text{the energy-momentum tensor of anisotropic matter}}
 \end{array}$$

where $x^i, i = 1, \dots, n$ are coordinates on spacetime M and $y^a, a = 1, 2, \dots, m$, are parameters (coordinates) of anisotropies. So, the Einstein theory formulated in the framework of (pseudo) Riemannian geometry has been considered to be locally isotropic. Usually, anisotropies are considered as induced by matter distributions, quantum fluctuations during inflation and so on. It is obvious that an energy-momentum tensor for a locally

anisotropic matter (depending on some anisotropy parameters) being proportional to the Einstein tensor must induce corresponding anisotropies of the metric if the theory is required to be self-consistent. This topic of construction of solutions of systems of Einstein–matter fields equations with both dependencies on anisotropic parameters of metric and matter fields is usually omitted in general relativity.

Metrics with local anisotropy (depending on tangent vectors) of type

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2(x^l, y^k)}{\partial y^i \partial y^j}$$

where

$$F(x^l, \lambda y^k) = \lambda F(x^l, y^k),$$

λ is a real number, i.e. the function $F(x^l, y^k)$ is homogeneous on variables y^k , were first considered by Finsler and usually they are associated to the so-called Finsler geometry provided with nonlinear and linear connections and metrics structures and respectively computed various types of curvatures and torsions (see references [9] and on further geometric generalizations and developments with applications in physics one could consult [19, 26–28, 32]). It was considered that such type geometries are less suitable for direct applications in physics because of substantial problems with definition of conservation laws (without local symmetries it was not clear in which manner one could define, for instance, energy momentum type values), absence of well backgrounded physical arguments and complexity of such type geometries.

Nevertheless, the very sceptic attitude was changed after it was proved that Finsler like geometries and their generalizations (the so called generalized Lagrange spaces [19]), as well physical theories with generic locally anisotropic interactions and/or Kaluza–Klein models, can be modelled in a unified geometrical manner on vector bundles provided with nonlinear connection structures (a subclass of anholonomic manifolds; as a general introduction into the spacetime differential geometry we cite [12] and on the geometry of nonlinear connections one could consult references [19] and [26, 27]).

The field equations of locally anisotropic gravity are of type

$$G_{\alpha\beta}(x^i, y^a) \simeq T_{\alpha\beta}(x^i, y^a)$$

where the Einstein tensor is defined on a bundle (generalized Finsler) space, x^i are usual coordinates on the base manifold and y^b are coordinates on the fibers (parameters of anisotropy). We emphasize that in general base and fiber spaces are of different dimensions, i.e. $\dim\{x^i\} \neq \dim\{y^b\}$. As a matter of principle this is a usual Einstein theory but on generic anholonomic manifolds (vector bundles) with a locally adapted nonlinear connection (equivalently, anisotropy) structure. Here we

also note that Finsler like metrics could be modelled in the framework of (pseudo)Riemannian geometry (under well defined conditions some Finsler metrics could be solutions of canonical Einstein equations) if dynamical reductions from higher dimensions to low dimensional ones are considered.

Since the seminal works of Bekenstein [4], Bardeen, Carter and Hawking [2] and Hawking [13], black holes were shown to have properties very similar to those of ordinary thermodynamics. One was treated the surface gravity on the event horizon as the temperature of the black hole and proved that a quarter of the event horizon area corresponds to the entropy of black holes. At present time it is widely believed that a black hole is a thermodynamic system (in spite of the fact that one have been developed a number of realizations of thermodynamics involving radiation) and the problem of statistical interpretation of the black hole entropy is one of the most fascinating subjects of modern investigations in gravitational and string theories.

In parallel to the ‘thermodynamilazation’ of black hole physics one have developed a new approach to the classical thermodynamics based of Riemannian geometry and its generalizations (a review on this subject is contained in Ref. [22]). Here is to be emphasized that geometrical methods have always played an important role in thermodynamics (see, for instance, a work by Blaschke [5] from 1923). Buchdahl used in 1966 a Euclidean metric in thermodynamics [6] and then Weinhold considered a sort of Riemannian metric [33]. It is considered that the Weinhold’s metric has not physical meaning in the context of purely equilibrium thermodynamics [21, 22] and Ruppeiner introduced a new metric (related via the temperature T as the conformal factor with the Weinhold’s metric).

The thermodynamical geometry was generalized in various directions, for instance, by Janyszek and Mrugała [15, 16, 20] even to discussions of applications of Finsler geometry in thermodynamic fluctuation theory and for nonequilibrium thermodynamics [24].

Our goal will be to provide a characterization of thermodynamics of $2+1$ -dimensional black la–holes with elliptical (constant in time) horizon obtained in [28]. From one point of view we shall consider the thermodynamic space of such objects (black la–holes in local equilibrium with la–spacetime ether) to depend on parameter of anisotropy, the angle θ , and on deformation parameter, the eccentricity ε . From another point, after we shall integrate the formulas on θ , the thermodynamic geometry will be considered in a usual two-dimensional Ruppeiner–Mrugała–Janyszek fashion. The main result of this work are the computation of thermodynamic curvatures and the proof that constant in time elliptic black la–holes have critical points of vanishing of curvatures (under both approaches to two-dimensional thermodynamic geometry) for some values of eccentricity, i. e. for under corresponding deformations of la–spacetimes.

The paper is organized as follows. In Sec. II, we

briefly review the geometry la-spacetimes provided with nonlinear connection structure and present the $2 + 1$ -dimensional constant in time elliptic black la-hole solution. In Sec. III, we state the thermodynamics of nearly equilibrium stationary black la-holes and establish the basic thermodynamic law and relations. In Sec. IV we develop two approaches to the thermodynamic geometry of black la-holes, compute thermodynamic curvatures and the equations for critical points of vanishing of curvatures for some values of eccentricity. In Sec. V, we draw a discussion and conclusions.

II. LOCALLY ANISOTROPIC SPACETIMES AND BLACK HOLES

In this section we outline for further applications the basic results on $2 + 1$ -dimensional la-spacetimes and black la-hole solutions (see Refs. [28,29] and the geometric background presented in Appendix; for $(2 + 1)$ -anisotropies all formula are considered for dimensions of base spaces $n = 3$ and of fibers $m = 1$).

A. Nonlinear Connections in $(2 + 1)$ -Dimensional Spacetimes

A $(2+1)$ -dimensional locally anisotropic spacetime is defined as a generic anholonomic vector bundle with an one dimensional fiber, parametrized by a coordinate y , over a two dimensional base space, locally parametrized by coordinates $x^i, i = 1, 2$. We shall use also the next denotations of coordinates $u = (x, y) = \{u^\alpha = (x^i, y)\}$.

The local anisotropy (la) is modelled by the coefficients of nonlinear connection (in brief, N-connection)

$$N = \{N_i(x, y) = N_i^b(x^j, y)\},$$

were, for simplicity, for N-connection components there are omitted the one dimensional fiber indices. Nonlinear connections generalize the concept of usual linear connections and can be treated as a field splitting the generic anholonomic spacetime into irreducible horizontal (base) and vertical (anisotropy) subspaces. The components of N-connection could be considered as prescribed values (functions) if some constraints on spacetime dynamics are imposed, or as coefficients of a specific nonlinear gauge field satisfying corresponding motion (field) equations. On generic la-spacetimes one have to apply 'elongated' by N-connections operators instead of usual local coordinate basis $\partial_\alpha = \partial/\partial u^\alpha$ and $d^\alpha = du^\alpha$, (see formulas (A3) and (A4)):

$$\begin{aligned} \delta_\alpha &= (\delta_i, \partial_{(y)}) = \frac{\partial}{\partial u^\alpha} \\ &= \left(\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i(x^j, y) \frac{\partial}{\partial y}, \partial_{(y)} = \frac{\partial}{\partial y} \right) \end{aligned} \quad (2.1)$$

and their duals

$$\begin{aligned} \delta^\beta &= (d^i, \delta^{(y)}) = \delta u^\beta \\ &= (d^i = dx^i, \delta^{(y)} = \delta y = dy + N_k(x^j, y) dx^k). \end{aligned} \quad (2.2)$$

The locally adapted to the N-connection structure operators of partial derivatives (2.1) and their duals, differentials, (2.2) form generic anholonomic frames, or la-bases. With respect to a fixed structure of la-bases and their tensor products we can construct distinguished, by N-connection, tensor algebras and various geometric objects (in brief, one writes d-tensors, d-metrics, d-connections and so on).

A symmetrical la-metric, or d-metric, could be written with respect to a la-basis (2.2) as

$$\begin{aligned} \delta s^2 &= g_{\alpha\beta}(u^\tau) \delta u^\alpha \delta u^\beta \\ &= g_{ij}(x^k, y) dx^i dx^j + h(x^k, y) (\delta y)^2. \end{aligned} \quad (2.3)$$

Such metrics have been used in generalized Finsler and Lagrange geometries [19] and for modelling Finsler-Kaluza-Klein (super)gravities on (super)vector bundles provided with N-connection structures [26–28].

B. Non-rotating black la-holes with ellipsoidal horizon

Let us consider a la-spacetime provided with local coordinates $x^1 = r$, $x^2 = \theta$ for the base subspace and when as the anisotropic direction is chosen the time like coordinate, $y = t$. We proved [29] that a d-metric of type (2.3),

$$\delta s^2 = g(r, \theta) dr^2 + r^2 d\theta^2 + h(r, \theta) \delta t^2, \quad (2.4)$$

where

$$\delta t = dt + N_1(r, \theta) dr + N_2(r, \theta) d\theta,$$

satisfies the system of vacuum la-gravitational equations in $(2+1)$ dimensions if

$$h(r, \theta) = \frac{1}{g(r, \theta)} = -\frac{p^2}{(1 + \varepsilon \cos(\theta - \theta_0))^2} + \frac{r^2}{r_0^2}, \quad (2.5)$$

where p, ε, θ_0 and r_0 are constants and the N-connection has the coefficients

$$\begin{aligned} N_1 &= H \frac{\partial g}{\partial r} \left[2r \frac{\partial g}{\partial r} + \left(\frac{\partial g}{\partial \theta} \right)^2 \right]^{-1}, \\ N_2 &= H \frac{\partial g}{\partial \theta} \left[2r \frac{\partial g}{\partial r} + \left(\frac{\partial g}{\partial \theta} \right)^2 \right]^{-1}, \end{aligned} \quad (2.6)$$

where

$$\frac{H}{2} = \frac{\partial^2 g}{\partial \theta^2} - \frac{1}{2g} \left(\frac{\partial g}{\partial \theta} \right)^2 - \frac{r}{g} \frac{\partial g}{\partial r}.$$

The time-time component $h(r, \theta)$ vanishes if

$$r_+^2 = \frac{p^2 r_0^2}{(1 + \varepsilon \cos(\theta - \theta_0))^2} \quad (2.7)$$

which is the square of the parametric equation of an ellipse with parameter p and eccentricity ε and where the angle θ_0 gives the orientation of axes. If we impose the condition that in the locally isotropic limit we shall have the usual BTZ solution, we can express the constants p and r_0 via the standard mass and cosmological constant, i. e. $p^2 = m_0$ and $r_0^2 = -1/\Lambda$. The eccentricity ε and axes orientation θ_0 are given by the initial conditions of la-gravitational space polarization.

We are thus led to the result that a d-metric (2.4), with coefficients (2.5) and for the N-connection (2.6), describes a locally anisotropic variant of the Schwarzschild metric and has an ellipsoidal horizon.

III. ON THE THERMODYNAMICS OF ELLIPTICAL BLACK LA-HOLES

In this paper we will be interested in thermodynamics of black la-holes defined by a d-metric (2.4).

The coefficient before r_0^2 in (2.7) can be treated as an anisotropic mass

$$m(\theta, \varepsilon) = \frac{m_0}{2\pi(1 + \varepsilon \cos(\theta - \theta_0))^2} = \frac{r_+^2}{2\pi r_0^2} \quad (3.1)$$

which depends on coordinate θ and eccentricity ε and on constants m_0 and θ_0 . The coefficient 2π was introduced in order to have the limit

$$\lim_{\varepsilon \rightarrow 0} 2 \int_0^\pi m(\theta, \varepsilon) d\theta = m_0. \quad (3.2)$$

Throughout this paper, the units $c = \hbar = k_B = 1$ will be used, but we shall consider that for an la-renormalized gravitational constant $8G_{(gr)}^{(a)} \neq 1$, see [29].

The Hawking temperature $T(\theta, \varepsilon)$ of a black la-hole is anisotropic and is computed by using the anisotropic mass (3.1):

$$T(\theta, \varepsilon) = \frac{m(\theta, \varepsilon)}{2\pi r_+(\theta, \varepsilon)} = \frac{r_+(\theta, \varepsilon)}{4\pi^2 r_0^2} > 0. \quad (3.3)$$

The two parametric analog of the Bekenstein-Hawking entropy is to be defined as

$$S(\theta, \varepsilon) = 4\pi r_+ = \sqrt{32\pi^3} |r_0| \sqrt{m(\theta, \varepsilon)} \quad (3.4)$$

The introduced thermodynamic quantities obey the first law of thermodynamics (under the supposition that the system is in local equilibrium under the variation of parameters (θ, ε))

$$\Delta m(\theta, \varepsilon) = T(\theta, \varepsilon) \Delta S, \quad (3.5)$$

where the variation of entropy is

$$\Delta S = 4\pi \Delta r_+ = 4\pi \frac{|r_0|}{\sqrt{m(\theta, \varepsilon)}} \left(\frac{\partial m}{\partial \theta} \Delta \theta + \frac{\partial m}{\partial \varepsilon} \Delta \varepsilon \right).$$

According to the formula $C = (\partial m / \partial T)$ we can compute the heat capacity

$$C = 2\pi r_+(\theta, \varepsilon) = 2\pi |r_0| \sqrt{m(\theta, \varepsilon)}.$$

Because of $C > 0$ always holds the temperature is increasing with the mass.

The formulas (3.1)-(3.5) can be integrated on angular variable θ in order to obtain some thermodynamic relations for black la-holes with elliptic horizon depending only on deformation parameter, the eccentricity ε .

For an elliptically deformed black la-hole with the outer horizon r_+ given by formula (3.4) the depending on eccentricity [29] Bekenstein-Hawking entropy is computed as

$$S^{(a)}(\varepsilon) = \frac{L_+}{4G_{(gr)}^{(a)}},$$

were

$$L_+(\varepsilon) = 4 \int_0^{\pi/2} r_+(\theta, \varepsilon) d\theta$$

is the length of ellipse's perimeter and $G_{(gr)}^{(a)}$ is the three dimensional gravitational coupling constant in la-media (the index (a) points to la-renormalizations), and has the value

$$S^{(a)}(\varepsilon) = \frac{2pr_0}{G_{(gr)}^{(a)} \sqrt{1 - \varepsilon^2}} \operatorname{arctg} \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}. \quad (3.6)$$

If the eccentricity vanishes, $\varepsilon = 0$, we obtain the locally isotropic formula with p being the radius of the horizon circumference, but the constant $G_{(gr)}^{(a)}$ could be la-renormalized.

The total mass of black la-hole of eccentricity ε is found by integrating (3.1) on angle θ :

$$m(\varepsilon) = \frac{m_0}{(1 - \varepsilon^2)^{3/2}} \quad (3.7)$$

which satisfies the condition (3.2).

The integrated on angular variable θ temperature $T(\varepsilon)$ is to be defined by using $T(\theta, \varepsilon)$ from (3.3),

$$T(\varepsilon) = 4 \int_0^{\pi/2} T(\theta, \varepsilon) d\theta = \frac{2\sqrt{m_0}}{\pi^2 |r_0| \sqrt{1 - \varepsilon^2}} \operatorname{arctg} \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}. \quad (3.8)$$

Formulas (3.6)-(3.8) describes the thermodynamics of ε -deformed black la-holes.

Finally, in this section, we note that a black la-hole with elliptic horizon is to be considered as a thermodynamic subsystem placed into the la-ether bath of space-time. To the la-ether one associates a continuous la-medium assumed to be in local equilibrium. The black la-hole subsystem is considered as a subsystem described by thermodynamic variables which are continuous field on variables (θ, ε) , or in the simplest case when one have integrated on θ , on ε . It will be our first task to establish some parametric thermodynamic relations between the mass $m(\theta, \varepsilon)$ (equivalently, the internal black la-hole energy), temperature $T(\theta, \varepsilon)$ and entropy $S(\theta, \varepsilon)$.

IV. THERMODYNAMIC METRICS AND CURVATURES OF BLACK LA-HOLES

We emphasize in this paper two approaches to the thermodynamic geometry of nearly equilibrium black la-holes based on their thermodynamics. The first one is to consider the thermodynamic space as depending locally on two parameters θ and ε and to compute the corresponding metric and curvature following standard formulas from curved bidimensional hypersurface Riemannian geometry. The second possibility is to take as basic the Ruppeiner metric in the thermodynamic space with coordinates (M, ε) , in a manner proposed in Ref. [7] with that difference that as the extensive coordinate is taken the black la-hole eccentricity ε (instead of the usual angular momentum J for isotropic $(2+1)$ -black holes). Of course, in this case we shall background our thermodynamic geometric constructions starting from the relations (3.6)–(3.8).

A. The thermodynamic parametric geometry

Let us consider the thermodynamic parametric geometry of the elliptic $(2+1)$ -dimensional black la-hole based on its thermodynamics given by formulas (3.1)–(3.5).

Rewriting equations (3.5), we have

$$\Delta S = \beta(\theta, \varepsilon) \Delta m(\theta, \varepsilon),$$

where $\beta(\theta, \varepsilon) = 1/T(\theta, \varepsilon)$ is the inverse to temperature (3.3). This case is quite different from that from [7,10] where there are considered, respectively, BTZ and dilaton black holes (by introducing Ruppeiner and Weinhold thermodynamic metrics). Our thermodynamic space is defined by a hypersurface given by parametric dependencies of mass and entropy. Having chosen as basic the relative entropy function,

$$\varsigma = \frac{S(\theta, \varepsilon)}{4\pi\sqrt{m_0}} = \frac{1}{1 + \varepsilon \cos \theta},$$

in the vicinity of a point $P = (0, 0)$, when, for simplicity, $\theta_0 = 0$, our hypersurface is given locally by conditions

$$\varsigma = \varsigma(\theta, \varepsilon)$$

and

$$\text{grad}_P \varsigma = 0.$$

For the components of bidimensional metric on the hypersurface we have

$$g_{11} = 1 + \left(\frac{\partial \varsigma}{\partial \theta} \right)^2, \quad g_{12} = \left(\frac{\partial \varsigma}{\partial \theta} \right) \left(\frac{\partial \varsigma}{\partial \varepsilon} \right),$$

$$g_{22} = 1 + \left(\frac{\partial \varsigma}{\partial \varepsilon} \right)^2,$$

The nonvanishing component of curvature tensor in the vicinity of the point $P = (0, 0)$ is

$$R_{1212} = \frac{\partial^2 \varsigma}{\partial \theta^2} \frac{\partial^2 \varsigma}{\partial \varepsilon^2} - \left(\frac{\partial^2 \varsigma}{\partial \theta \partial \varepsilon} \right)^2$$

and the curvature scalar is

$$R = 2R_{1212}. \quad (4.1)$$

By straightforward calculations we can find the condition of vanishing of the curvature (4.1) when

$$\varepsilon_{\pm} = \frac{-1 \pm (2 - \cos^2 \theta)}{\cos \theta (3 - \cos^2 \theta)}. \quad (4.2)$$

So, the parametric space is separated in subregions with elliptic eccentricities $0 < \varepsilon_{\pm} < 0$ and θ satisfying conditions (4.2).

Ruppeiner suggested that the curvature of thermodynamic space is a measure of the smallest volume where classical thermodynamic theory based on the assumption of a uniform environment could conceivably work and that near the critical point it is expected this volume to be proportional to the scalar curvature [22]. There were also proposed geometric equations relating the thermodynamic curvature via inverse relations to free energy. Our definition of thermodynamic metric and curvature in parametric spaces differs from that of Ruppeiner or Weinhold and it is obvious that relations of type (4.2) (stating the conditions of vanishing of curvature) could be related with some conditions for stability of thermodynamic space under variations of eccentricity ε and anisotropy angle θ . This interpretation is very similar to that proposed by Janyszek and Mrugala [14] and supports the viewpoint that the first law of thermodynamics makes a statement about the first derivatives of the entropy, the second law is for the second derivatives and the curvature is a statement about the third derivatives. This treatment holds good also for the parametric thermodynamic spaces for black la-holes.

B. Thermodynamic Metrics and Eccentricity of Black La-Hole

A variant of thermodynamic geometry of black la-holes could be backgrounded on integrated on anisotropy angle

θ formulas (3.6)-(3.8). The Ruppeiner metric of elliptic black la-holes in coordinates (m, ε) is

$$ds_R^2 = - \left(\frac{\partial^2 S}{\partial m^2} \right)_\varepsilon dm^2 - \left(\frac{\partial^2 S}{\partial \varepsilon^2} \right)_m d\varepsilon^2. \quad (4.3)$$

For our further analysis we shall use dimensionless values $\mu = m(\varepsilon)/m_0$ and $\zeta = S^{(a)}G_{gr}^{(a)}/2pr_0$ and consider instead of (4.3) the thermodynamic diagonal metrics $g_{ij}(a^1, a^2) = g_{ij}(\mu, \varepsilon)$ with components

$$g_{11} = - \frac{\partial^2 \zeta}{\partial \mu^2} = -\zeta_{,11} \quad \text{and} \quad g_{22} = - \frac{\partial^2 \zeta}{\partial \varepsilon^2} = -\zeta_{,22}, \quad (4.4)$$

where by comas we have denoted partial derivatives.

The expressions (3.6) and (3.7) are correspondingly rewritten as

$$\zeta = \frac{1}{\sqrt{1-\varepsilon^2}} \operatorname{arctg} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$$

and

$$\mu = (1 - \varepsilon^2)^{-3/2}.$$

By straightforward calculations we obtain

$$\begin{aligned} \zeta_{,11} = & -\frac{1}{9} (1 - \varepsilon^2)^{5/2} \operatorname{arctg} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \\ & + \frac{1}{9\varepsilon} (1 - \varepsilon^2)^3 + \frac{1}{18\varepsilon^4} (1 - \varepsilon^2)^4 \end{aligned}$$

and

$$\zeta_{,22} = \frac{1 + 2\varepsilon^2}{(1 - \varepsilon^2)^{5/2}} \operatorname{arctg} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} - \frac{3\varepsilon}{(1 - \varepsilon^2)^2}.$$

The thermodynamic curvature of metrics of type (4.4) can be written in terms of second and third derivatives [14] by using third and second order determinants:

$$\begin{aligned} R = & \frac{1}{2} \begin{vmatrix} -\zeta_{,11} & 0 & -\zeta_{,22} \\ -\zeta_{,111} & -\zeta_{,112} & 0 \\ -\zeta_{,112} & 0 & -\zeta_{,222} \end{vmatrix} \times \begin{vmatrix} -\zeta_{,11} & 0 \\ 0 & -\zeta_{,22} \end{vmatrix}^{-2} \\ = & -\frac{1}{2} \left(\frac{1}{\zeta_{,11}} \right)_{,2} \times \left(\frac{\zeta_{,11}}{\zeta_{,22}} \right)_{,2}. \end{aligned} \quad (4.5)$$

The conditions of vanishing of thermodynamic curvature (4.5) are as follows

$$\zeta_{,112}(\varepsilon_1) = 0 \quad (4.6)$$

or

$$\left(\frac{\zeta_{,11}}{\zeta_{,22}} \right)_{,2}(\varepsilon_2) = 0 \quad (4.7)$$

for some values of eccentricity, $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$, satisfying conditions $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$. For small deformations of black la-holes, i.e. for small values of eccentricity, we can approximate $\varepsilon_1 \approx 1/\sqrt{5.5}$ and $\varepsilon_2 \approx 1/(18\lambda)$,

where λ is a constant for which $\zeta_{,11} = \lambda \zeta_{,22}$ and the condition $0 < \varepsilon_2 < 1$ is satisfied. We omit general formulas for curvature (4.5) and conditions (4.6) and (4.7), when the critical points ε_1 and/or ε_2 must be defined from nonlinear equations containing $\operatorname{arctg} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$ and powers of $(1 - \varepsilon^2)$ and ε .

V. DISCUSSION AND CONCLUSIONS

In closing, we would like to discuss the meaning of geometric thermodynamics following from black locally anisotropic (la) holes (in brief we shall use la-holes, la-spacetime and so on).

(1) *Nonequilibrium thermodynamics of black la-holes in la-spacetimes.* In this paper and in paper I [29] we concluded that the thermodynamics of generic la-spacetimes has a generic nonequilibrium character and could be developed in a geometric fashion following the approach proposed by S. Sieniutycz, P. Salamon and R. S. Berry [24,23]. This could form a new branch of black hole thermodynamics which should be based on previous studies in nonequilibrium thermodynamics and kinetics, their stochastic [31] and statistical background, and on thermodynamical hydrodynamics. All constructions will be performed on generic anholonomic manifolds by modelling la-spacetimes on vector bundles provided with nonlinear connection structure. This direction could be motivated after it was found that generalized and standard Finsler like metrics could be considered in the framework of Einstein gravity under corresponding parametrizations of metrics and reductions from higher dimensions to low dimensional ones.

(2) *Black la-holes thermodynamics in vicinity of equilibrium points.* The usual thermodynamical approach in the Bekenstein-Hawking manner is valid for black la-holes for a subclass of such physical systems when the hypothesis of local equilibrium is physically motivated and corresponding renormalizations, by la-spacetime parameters, of thermodynamical values are defined.

(3) *The geometric thermodynamics of black la-holes with constant in time elliptic horizon* was formulated following two approaches: for a parametric thermodynamic space depending on anisotropy angle θ and eccentricity ε and in a standard Ruppeiner-Mrugala-Janyszek fashion, after integration on anisotropy θ but maintaining la-spacetime deformations on ε .

(4) *The thermodynamic curvatures of black la-holes* were shown to have critical values of eccentricity when the scalar curvature vanishes. Such type of thermodynamical systems are rather unusual and a corresponding statistical model is not that for ordinary systems composed by classical or quantum like gases.

(5) *Thermodynamic systems with constraints* requires a new geometric structure in addition to the thermodynamical metrics which is that of nonlinear connection.

We note this object must be introduced both in spacetime geometry and in thermodynamic geometry if generic anisotropies and constrained field and/or thermodynamic behavior are analyzed.

APPENDIX A: ANHOLONOMIC BUNDLES AND LOCALLY ANISOTROPIC SPACES

We outline the geometry of anholonomic bundles and of the N-connection structure for locally anisotropic spaces. Details on spacetime differential geometry and on nonlinear connections are contained respectively in [12] and [19]

1. Anholonomic manifolds and gravity

In this section all manifolds $E_{d(E)}$ are assumed to be smooth (i.e. C^∞), of finite integer dimension $d(E) \geq 3, 4, \dots$, Hausdorff, paracompact and connected; all maps are smooth. We denote the local coordinates on $E_{d(E)}$ by variables u^α , where Greek indices takes values $\alpha, \beta, \dots = 1, 2, 3, 4, \dots$ and could be both type coordinate or abstract (Penrose's) ones. A spacetime is modelled by a manifold $E_{d(E)}$ provided with corresponding geometric structures (symmetric metric $g_{\alpha\beta}$; linear, in general nonsymmetric, connection $\Gamma_{\beta\gamma}^\alpha$ defining the covariant derivation ∇_α ; nonmetricity $Q_{\alpha\beta\gamma} = \nabla_\alpha g_{\beta\gamma}$ which in this work is considered to be vanishing, i.e. $Q_{\alpha\beta\gamma} \equiv 0$). If it would be necessary to emphasize that some indices are abstract marks, we shall underline them, i.e. we shall write $\underline{\alpha}, \underline{\beta}, \dots$, or put them into round brackets, $(\underline{\alpha}), (\underline{\beta}), \dots$

Frame basis vectors on $E_{d(E)}$, numbered by a index $\underline{\alpha}$ are denoted by $e_{\underline{\alpha}}$ with components $e_{\underline{\alpha}}^\alpha = g_{\alpha\beta} e_{\underline{\alpha}}^{\alpha\beta}$, i.e. $e_{\underline{\alpha}} = \{e_{\underline{\alpha}}^\alpha = g_{\alpha\beta} e_{\underline{\alpha}}^{\alpha\beta}\}$, and they are subjected to relations

$$e_{\underline{\alpha}} e_{\underline{\beta}}^\alpha = \eta_{\underline{\alpha}, \underline{\beta}}, \quad \text{and} \quad e_{\underline{\alpha}}^\alpha e_{\underline{\beta}}^{\alpha\beta} \eta_{\underline{\alpha}, \underline{\beta}} = g_{\alpha\beta} \quad (A1)$$

where the Einstein summations rule is accepted, $\eta_{\underline{\alpha}, \underline{\beta}}$ is a given constant symmetric matrix of signature $(-, +, \dots, +)$ (the sign minus is used in this work for the time coordinate of spacetime). Operations with underlined and non-underlined indices are correspondingly performed by using the matrix $\eta_{\underline{\alpha}, \underline{\beta}}$, its inverse $\eta^{\underline{\alpha}\underline{\beta}}$, and the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. A frame basis structure on $E_{d(E)}$ is characterized by its anholonomy coefficients $w_{\beta\gamma}^\alpha$ defined from relations

$$e_\alpha e_\beta - e_\beta e_\alpha = w_{\alpha\beta}^\gamma e_\gamma. \quad (A2)$$

With respect to a fixed basis e_α and its dual e^β on $E_{d(E)}$ we can decompose tensors and define their components, for instance,

$$T = T_{\alpha\beta}^\gamma e_\gamma \otimes e^\alpha \otimes e^\beta$$

where by \otimes it is denoted the tensor product.

We can also consider local linear transforms of frames, $e_{\underline{\alpha}} = a_{\underline{\alpha}}^{\alpha'} e_{\alpha'}$, parametrized by nondegenerated linear matrices $a_{\underline{\alpha}}^{\alpha'}$ and define the corresponding linear frame bundle on $\bar{E}_{d(E)}$, or introduce local affine transforms of frames, $e_{\underline{\alpha}} = a_{\underline{\alpha}}^{\alpha'} e_{\alpha'} + q_{\underline{\alpha}}$, with additional affine shifts given by $q_{\underline{\alpha}}$ and define the corresponding affine frame bundle on $\bar{E}_{d(E)}$.

Now we discuss the difference between the anholonomic and holonomic manifolds. A spacetime $E_{d(E)}$ is **holonomic (locally integrable)** if it admits a frame structure for which the anholonomy coefficients from (A2) vanishes, i.e. $w_{\alpha\beta}^\gamma = 0$. In this case we can introduce local coordinate bases,

$$\partial_\alpha = \partial / \partial u^\alpha \quad (A3)$$

and their duals

$$d^\alpha = du^\alpha \quad (A4)$$

and consider decompositions of geometrical objects with respect to such frames. The general relativity theory was formally defined for holonomic pseudo-Riemannian manifold. For various purposes on holonomic spacetimes it is convenient to use anholonomic frames $e_{\underline{\alpha}}$ satisfying conditions (A2), but we emphasize that for such spaces we can always define linear transforms of frames to a coordinate basis of type (A3), $e_{\underline{\alpha}} = a_{\underline{\alpha}}^{\alpha'} \partial_{\alpha'}$. By applying frames and theirs transforms on holonomic pseudo-Riemannian spaces there were developed the so-called tetradic and spinor gravity and extensions to linear, affine and de Sitter gauge group gravity models.

A spacetime $E_{d(E)}$ is **anholonomic (locally non-integrable)** if it does not admit a frame structure for which the anholonomy coefficients from (A1) vanishes, i.e. $w_{\alpha\beta}^\gamma \neq 0$. In this case the anholonomy becomes a generic spacetime characteristics. It induces nonvanishing additional terms into the torsion, $T(\delta_\gamma, \delta_\beta) = T_{\beta\gamma}^\alpha \delta_\alpha$, and curvature, $R(\delta_\tau, \delta_\gamma) \delta_\beta = R_{\beta\gamma\tau}^\alpha \delta_\alpha$, tensors of a linear connection $\Gamma_{\beta\gamma}^\alpha$, with coefficients defined respectively as

$$T_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha + w_{\beta\gamma}^\alpha \quad (A5)$$

and

$$R_{\beta\gamma\tau}^\alpha = \delta_\tau \Gamma_{\beta\gamma}^\alpha - \delta_\gamma \Gamma_{\beta\tau}^\alpha + \Gamma_{\beta\gamma}^\varphi \Gamma_{\varphi\tau}^\alpha - \Gamma_{\beta\tau}^\varphi \Gamma_{\varphi\gamma}^\alpha + \Gamma_{\beta\varphi}^\alpha w_{\gamma\tau}^\varphi. \quad (A6)$$

The Ricci tensor is defined as

$$R_{\beta\gamma} = R_{\beta\gamma\alpha}^\alpha \quad (A7)$$

and the scalar curvature is

$$R = g^{\beta\gamma} R_{\beta\gamma} \quad (A8)$$

The Einstein equations on a anholonomic spacetime are introduced in a standard manner:

$$R_{\beta\gamma} - \frac{1}{2}g_{\beta\gamma}R = k\Upsilon_{\beta\gamma}, \quad (A9)$$

where the energy-momentum d-tensor $\Upsilon_{\beta\gamma}$ includes the cosmological constant terms and possible contributions of torsion (A5) and matter and k is the coupling constant. For symmetric linear connection the torsion field can be considered as induced by the anholonomy (or equivalently, by imposed constraints). For dynamical torsions there are necessary additional field equations, see, for instance, the case of gauge like theories [32].

The usual locally isotropic Einstein gravity is obtained under suppositions that we restrict our considerations only for frame fields (for a four dimensional spacetime having 16 components) which are locally isotropic and are locally linearly equivalent to a coordinate vector basis and could generate via relations (A1) the pseudo-Riemannian metric (having 10 components for a 4 dimensional spacetime).

A subclass of anholonomic spacetimes consists from those with local anisotropy. It is a matter of further theoretical and experimental investigations in order to establish if the present day experimental data on anisotropic structure of Universe is a consequence of matter and quantum fluctuation induced anisotropies and for some scales being a consequence of anholonomy of observator's frame, both cases being considered for a locally isotropic spacetime background, or the spacetime anisotropy is a generic property following, for instance, from string theory, and from a more general self-consistent gravitational theory when both the left (geometric) and right (matter energy-momentum tensor) parts of Einstein equations depends on anisotropic parameters.

2. The local anisotropy and nonlinear connection

In this subsection we briefly outline the geometry of locally anisotropic spaces.

Roughly, a local anisotropy is introduced by calling some spacetime directions (coordinates) to be anisotropic. In this case the spacetime dimension is split locally into two components, n for isotropic coordinates and m for anisotropic coordinates, when $d_{(E)} = n + m$ with $n \geq 2$ and $m \geq 1$. We shall use local coordinates $u^\alpha = (x^i, y^a)$, where Greek indices α, β, \dots take values $1, 2, \dots, n + m$ and Latin indices i and a are correspondingly n and m dimensional, i.e. $i, j, k, \dots = 1, 2, \dots, n$ and $a, b, c, \dots = 1, 2, \dots, m$.

There is necessary a correct geometric definition of decomposition of spacetime into isotropic and anisotropic components. For modelling of a locally anisotropic spacetime, in brief a la-space, we choose a vector bundle, $\mathcal{E} = (E_{n+m}, p, M_n, F_m, Gr)$ provided with **nonlinear connection (in brief N-connection)** structure $N =$

$\{N_j^a(u^\alpha)\}$, where $N_j^a(u^\alpha)$ are its coefficients. We use denotations: E_{n+m} for the $(n+m)$ -dimensional total space of the vector bundle; M_n for the n -dimensional base manifold; F_m for the typical fiber being a m -dimensional real vector space; Gr is the group of automorphisms of F_m and p is a surjective mapping. For simplicity we shall consider only local constructions on vector bundles.

The N-connection is a new geometric object which generalize that of linear connection. This concept came from Finsler geometry (see the Cartan's monograph [8]), the global formulation of it is due to W. Barthel [3], and it is studied in details in Miron and Anastasiei works [19]. We have extended the geometric constructions for spinor bundles and superbundles with further applications in locally anisotropic field theory and strings.

The rigorous mathematical definition of N-connection is based on the formalism of horizontal and vertical subbundles and on exact sequences of vector bundles. Here, for simplicity, we define a N-connection as a distribution which for every point $u = (x, y) \in \mathcal{E}$ defines a local decomposition of the tangent space of our vector bundle, $T_u E$, into horizontal, $H_u E$, and vertical (anisotropy), $V_u E$, directions, i.e.

$$T_u E = H_u E \oplus V_u E.$$

If a N-connection with coefficients $N_j^a(u^\alpha)$ is introduced on the vector bundle \mathcal{E} the modelled spacetime posses a generic local anisotropy and in this case we can not even apply in a usual manner the partial derivatives and their duals, differentials. Instead of coordinate bases (A3) and (A4) we must consider some bases adapted to the N-connection structure:

$$\begin{aligned} \delta_\alpha &= (\delta_i, \partial_a) = \frac{\delta}{\partial u^\alpha} \\ &= \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^b(x^j, y) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a} \right) \end{aligned} \quad (A10)$$

and

$$\begin{aligned} d^\beta &= (d^i, \delta^a) = \delta u^\beta \\ &= (d^i = dx^i, \delta^a = dy^a = dy^a + N_k^a(x^j, y^b) dx^k). \end{aligned} \quad (A11)$$

A nonlinear connection (N-connection) is characterized by its curvature

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (A12)$$

The elongation (by N-connection) of partial derivatives, from (A10), called the adapted to the N-connection partial derivatives, or the locally adapted basis (la-basis) δ_β , reflects the fact that the spacetime is locally anisotropic \mathcal{E} and generically anholonomic because there are satisfied anholonomy relations (A2),

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w_{\alpha\beta}^\gamma \delta_\gamma,$$

where anolonomy coefficients are as follows

$$w_{ij}^k = 0, w_{aj}^k = 0, w_{ia}^k = 0, w_{ab}^k = 0, w_{ab}^c = 0, \\ w_{ij}^a = -\Omega_{ij}^a, w_{aj}^b = -\partial_a N_i^b, w_{ia}^b = \partial_a N_i^b.$$

On a la-space the geometrical objects have a distinguished (by N-connection), into horizontal and vertical components, character. They are briefly called d-tensors, d-metrics and/or d-connections. Their components are defined with respect to a la-basis of type (A10), its dual (A11), or their tensor products (d-linear or d-affine transforms of such frames could also be considered). For instance a covariant and contravariant d-tensor Z , is expressed as

$$Z = Z_{\beta}^{\alpha} \delta_{\alpha} \otimes \delta^{\beta} \\ = Z_{ij}^i \delta_i \otimes d^j + Z_a^i \delta_i \otimes \delta^a + Z_j^b \partial_b \otimes d^j + Z_a^b \partial_b \otimes \delta^a.$$

A symmetric d-metric on la-space \mathcal{E} is written as

$$\delta s^2 = g_{\alpha\beta}(u) \delta^{\alpha} \otimes \delta^{\beta} \\ = g_{ij}(x, y) dx^i dx^j + h_{ab}(x, y) \delta y^a \delta y^b. \quad (A13)$$

A linear d-connection D on la-space \mathcal{E} ,

$$D_{\delta_{\gamma}} \delta_{\beta} = \Gamma_{\beta\gamma}^{\alpha} (x^k, y) \delta_{\alpha},$$

is parametrized by non-trivial h-v-components,

$$\Gamma_{\beta\gamma}^{\alpha} = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a) \quad (A14)$$

Some d-connection and d-metric structures are compatible if there are satisfied the conditions

$$D_{\alpha} g_{\beta\gamma} = 0.$$

For instance, a canonical compatible d-connection

$${}^c\Gamma_{\beta\gamma}^{\alpha} = ({}^cL_{jk}^i, {}^cL_{bk}^a, {}^cC_{jc}^i, {}^cC_{bc}^a)$$

is defined by the coefficients of d-metric (A13), $g_{ij}(x, y)$ and $h_{ab}(x, y)$, and by the coefficients of N-connection,

$${}^cL_{jk}^i = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}), \\ {}^cL_{bk}^a = \partial_b N_k^a + \frac{1}{2} h^{ac} (\delta_k h_{bc} - h_{dc} \partial_b N_i^d - h_{db} \partial_c N_i^d), \\ {}^cC_{jc}^i = \frac{1}{2} g^{ik} \partial_c g_{jk}, \\ {}^cC_{bc}^a = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc})$$

This d-connection generalizes for la-spaces the well known Cristoffel symbols.

For a d-connection (A14) we can compute the components of, in our case d-torsion, (A5):

$$T_{jk}^i = T_{jk}^i = L_{jk}^i - L_{kj}^i, \quad T_{ja}^i = C_{ja}^i, T_{aj}^i = -C_{ja}^i, \\ T_{ja}^i = 0, \quad T_{bc}^a = S_{bc}^a = C_{bc}^a - C_{cb}^a, \\ T_{ij}^a = -\Omega_{ij}^a, \quad T_{bi}^a = \partial_b N_i^a - L_{bj}^a, \quad T_{ib}^a = -T_{bi}^a.$$

In a similar manner, putting non-vanishing coefficients (A14) into the formula for curvature (A6), we can compute the non-trivial components of d-curvature

$$R_{h.jk}^i = \delta_k L_{hj}^i - \delta_j L_{hk}^i \\ + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{jk}^a, \\ R_{b.jk}^a = \delta_k L_{bj}^a - \delta_j L_{bk}^a \\ + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a \Omega_{jk}^c, \\ P_{j.ka}^i = \partial_k L_{jk}^i + C_{jb}^i T_{ka}^b \\ - (\partial_k C_{ja}^i + L_{lk}^i C_{ja}^l - L_{jk}^l C_{la}^i - L_{ak}^c C_{jc}^i), \\ P_{b.ka}^c = \partial_a L_{bk}^c + C_{bd}^c T_{ka}^d \\ - (\partial_k C_{ba}^c + L_{dk}^c C_{ba}^d - L_{bk}^d C_{da}^c - L_{ak}^d C_{bd}^c), \\ S_{j.bc}^i = \partial_c C_{jb}^i - \partial_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i, \\ S_{b.cd}^a = \partial_d C_{bc}^a - \partial_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a.$$

The components of the Ricci tensor tensor (A7)

$$R_{\alpha\beta} = R_{\alpha,\beta\tau}^{\tau}$$

with respect to locally adapted frames (A9) and (A10) (in our case, d-tensor) are as follows:

$$R_{ij} = R_{i,jk}^k, \quad R_{ia} = -{}^2P_{ia} = -P_{i,ka}^k, \quad (A15) \\ R_{ai} = {}^1P_{ai} = P_{a,ib}^b, \quad R_{ab} = S_{a,bc}^c.$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d-tensor is non symmetric.

Having defined a d-metric of type (A13) in \mathcal{E} we can introduce the scalar curvature (A8) of a d-connection D ,

$$\overline{R} = G^{\alpha\beta} R_{\alpha\beta} = \widehat{R} + S, \quad (A16)$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

Now, by introducing the values (A15) and (A16) into anholonomic gravity field equations (A9) we can write down the system of Einstein equations for la-gravity with prescribed N-connection structure [19]:

$$R_{ij} - \frac{1}{2} (\widehat{R} + S) g_{ij} = k \Upsilon_{ij}, \quad (A17) \\ S_{ab} - \frac{1}{2} (\widehat{R} + S) h_{ab} = k \Upsilon_{ab}, \\ {}^1P_{ai} = k \Upsilon_{ai}, \\ {}^2P_{ia} = -k \Upsilon_{ia},$$

where $\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}$ and Υ_{ia} are the components of the energy-momentum d-tensor field.

There are variants of la-gravitational field equations derived in the low-energy limits of the theory of locally anisotropic (super)strings [27] or in the framework of gauge like la-gravity [32,28] when the N-connection and torsions are dynamical fields.

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